Advanced Topics in Random Graphs
Exercise Sheet 1

Question 1. Let $H = (V, E)$ be a hypergraph in which every edge has at least k elements, and suppose that each each intersects at most d other edges.

Show that, if $e(d + 1) \leq 2^{k-1}$, then we can 2-colour V such that no edge is monochromatic.

A hypergraph is k-uniform if every edge has size k and k-regular if every v lies in k edges. Deduce that the same holds true for any k-uniform, k-regular hypergraph, as long as $k \geq 9$.

Question 2. Suppose we have some collection of variables x_i which can take the values 0 or 1. We are given some set of statements of the form

$$S_i = x_{i_1} \vee \neg x_{i_2} \vee \ldots \vee x_{i_k}$$

all involving k variables which can either be x_i or $\neg x_i$ for some i. We wish to find some assignment for the variables such that all of the statements are true (that is, equal to 1).

Suppose we have a set of statements such that each x_i (or it's negation) appears in at most $2^{k-2}/k$ statements, show that we can find such an assignment.

Question 3. Suppose we are given n pairs of points in some graph G, and for each pair x_i, y_i a collection F_i of a least m paths betweens x_i and y_i. Suppose further that for every i and j, each path in F_i shares an edge with at most k paths in F_j.

Show that if $m \geq 6nk$ it is possible to find a disjoint family of paths joining the pairs together.

Question 4. Let $D = (V, E)$ be a directed graph with minimum outdegree δ and maximum indegree Δ. Suppose $k \in \mathbb{N}$ is such that

$$k \leq \frac{\delta}{1 + \log(1 + \delta\Delta)},$$

show that D contains a directed cycle of length at least k.

Question 5. Let G be a d-regular graph with girth at least 6. Show that we can colour the vertices of G with $cd^{\frac{4}{3}}$ colours, for some c sufficiently large, such that no cycle is 2-coloured (not insisting the the colouring is proper).

(Hint: Consider the set of events, for each path with 4 edges, that the path is 2-coloured.)

Show further that we can find such a colouring which is proper.